Neutral curves and stability boundaries in stratified flow

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An example is presented which shows that the boundary of stability in an antisymmetric stratified shear flow is not necessarily marked by steady neutral waves with $c = 0$. The stability characteristics of stratified shear flow in the neighbourhood of the neutral curve are also discussed.

1. Introduction

This note is intended as a supplementary comment on a recent paper by Miles (1963). Professor Miles has suggested to me that an example I constructed last summer may be of some general interest. This example seems to be the simplest one illustrating the fact that the 'principle of exchange of stabilities' is not necessarily true for antisymmetric stratified shear flow, and also helps to clarify the physical mechanism which in some cases leads to the violation of this principle. This example is presented and discussed in § 2.

I also take this opportunity to present, in § 3, a generalization, to the case of stratified flow, of the well-known perturbation for the unstable waves adjacent to the neutral solution corresponding to the inflexion point, in homogeneous parallel inviscid flow (cf. Lin 1955). This gives a way of finding some aspects of the stability of stratified flow near the neutral curve which seems to be rather more direct than the methods used in Miles's paper (1963) and does not appear to require the analyticity of the velocity and density profiles. On the other hand, Miles's methods give a more detailed picture of the instability, and seem more suitable for studying higher approximations.

2. Example

In this note, as in Miles (1963), the 'Boussinesq approximation' of neglect of the inertial effects of density stratification is used. The stability equation then becomes (cf. Miles 1961):

$$[(U-c)^2 F']' - \alpha^2 (U-c)^2 F + g\beta F = 0, \tag{1}$$

where $U = U(y)$ is the basic parallel flow, the stream function of the perturbation is $(U-c)\,F(y)\,e^{i\alpha(x-ct)}$ and $\beta(y) = -\rho'/\rho$ is the static stability. If U is odd and β is even, this equation is unchanged if c is replaced by $-c$ and $F(y)$ by $F(-y)$. We suppose that the boundaries, if any (on which the boundary conditions $F = 0$ are imposed), are also symmetrically disposed with respect to $y = 0$. The equation is also unchanged if c is replaced by its complex conjugate $\bar{c}$ and F by $\bar{F}$. Thus an eigenfunction with $c = c_r + ic_i$ must be accompanied by eigenfunctions with

$c = \pm c_r \pm ic_i$, the two ambiguous signs being independent, though only the solutions with $c_i \geqslant 0$ are actually relevant to the solution of an initial-value problem, or as limits for viscosity approaching zero. In particular, a singular neutral mode, i.e. a solution with $c_i = 0$ which is a limit of unstable solutions, must have $c = 0$ ('principle of exchange of stabilities') if it is *unique*; similarly one might expect an unstable wave to have c pure imaginary. Physically speaking, the symmetry (or antisymmetry) means that neither left nor right is preferred as a direction of propagation of the wave, so the wave must stand still—*unless 'it' is actually two waves with opposite directions of propagation* thus restoring the symmetry without requiring $c = 0$ or c pure imaginary. The purpose of the present example is to show that this second possibility can really occur, even with monotonic density and velocity profiles.

The example is

$$\left.\begin{aligned} U &= \operatorname{sgn} y \quad (|y| > 1), \\ U &= 0 \qquad\;\; (|y| < 1), \\ \beta &= \delta(y+1) + \delta(y-1), \end{aligned}\right\} \tag{2}$$

i.e. essentially a pair of Kelvin–Helmholtz shear layers. Within the intervals of continuity of the profiles, the stability equation reduces to $F'' - \alpha^2 F = 0$; the boundary conditions are $F(\pm\infty) = 0$, and the jump conditions at ± 1, derived for example by the usual considerations of continuity of pressure, etc., are: F continuous, $\Delta[(U-c)^2 F'] + gF = 0$, Δ standing for the jump in crossing the interface in the positive sense. The velocity and length scales having been chosen already in a dimensionless way in writing down (2), the parameter g should be thought of as an overall Richardson number, rather than the gravitational acceleration. By using the boundary and jump conditions, the eigenfunction is found to be:

$$\left.\begin{aligned} F &= (A + B\,e^{-2\alpha})\,e^{\alpha(y+1)} \quad (y < -1) \\ &= A\,e^{-\alpha(y+1)} + B\,e^{\alpha(y-1)} \quad (-1 < y < 1) \\ &= (A\,e^{-2\alpha} + B)\,e^{-\alpha(y-1)} \quad (1 < y), \end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned} &A[-1+g/\alpha-2c-2c^2] + B\,e^{-2\alpha}[-1+g/\alpha-2c] = 0 \\ \text{and}\quad &A\,e^{-2\alpha}[-1+g/\alpha+2c] + B[-1+g/\alpha+2c-2c^2] = 0. \end{aligned}\right\} \tag{4}$$

The eigenvalue relation is obtained by insuring that the two equations (4) are consistent; setting $h = g/\alpha - 1$ and $a^2 = 1 - e^{-4\alpha}$ it can be written

$$[2c^2 - h]^2 - 4c^2a^2 = (1-a^2)\,h^2. \tag{5}$$

The temptation to solve this as a quadratic in c^2 should be resisted, and (5) rewritten as

$$[2c^2 - h - a^2]^2 = (a^2+h)^2 - a^2h^2 = (a^2+h-ah)\,(a^2+h+ah). \tag{6}$$

Let $R_1^2 = \frac{1}{4}(a^2+h-ah)$ and $R_2^2 = \frac{1}{4}(a^2+h+ah)$. Then (6) becomes

$$(c^2 - R_1^2 - R_2^2)^2 = 4R_1^2R_2^2, \quad\text{or}\quad [c^2 - (R_1+R_2)^2]\,[c^2 - (R_1-R_2)^2] = 0.$$

From this we see that the four roots of (5) are all given by

$$c = R_1 + R_2, \tag{7}$$

where $R_1^2 = \frac{1}{4}(a^2+h-ah)$, $R_2^2 = \frac{1}{4}(a^2+h+ah)$, the two independent ambiguous signs implied in the definitions of R_1 and R_2 giving the four roots. The stability characteristics are now easily read off. Since $a < 1$, it is clear that both R_1^2 and R_2^2 are positive when

$$h > -a^2/(1+a),$$

$R_2^2 < 0$ and $R_1^2 > 0$ when

$$-a^2/(1-a) < h < -a^2/(1+a),$$

and both are negative when $h < -a^2/(1-a)$. Returning to the original variables we have:

(*a*) *Stability*, with four real roots if

$$\frac{g}{\alpha} > \frac{e^{-4\alpha}+(1-e^{-4\alpha})^{\frac{1}{2}}}{1+(1-e^{-4\alpha})^{\frac{1}{2}}}.$$

(*b*) *Instability*, with two pairs of conjugate complex (not pure imaginary) roots if

$$\frac{e^{-4\alpha}-(1-e^{-4\alpha})^{\frac{1}{2}}}{1-(1-e^{-4\alpha})^{\frac{1}{2}}} < \frac{g}{\alpha} < \frac{e^{-4\alpha}+(1-e^{-4\alpha})^{\frac{1}{2}}}{1+(1-e^{-4\alpha})^{\frac{1}{2}}}.$$

(*c*) *Instability*, with two pairs of conjugate pure imaginary roots if

$$\frac{g}{\alpha} < \frac{e^{-4\alpha}-(1-e^{-4\alpha})^{\frac{1}{2}}}{1-(1-e^{-4\alpha})^{\frac{1}{2}}}$$

(this last case can occur with $g > 0$ only if

$$\alpha < \alpha_* = \tfrac{1}{4}\log\tfrac{1}{2}(\sqrt{5}+1) \simeq 0{\cdot}1203).$$

Thus the stability boundary is

$$\frac{g}{\alpha} = \frac{e^{-4\alpha}+(1-e^{-4\alpha})^{\frac{1}{2}}}{1+(1-e^{-4\alpha})^{\frac{1}{2}}} \tag{8}$$

and is not a locus of $c = 0$; on this neutral curve we have in fact

$$c = \pm a^2[2a(1+a)]^{-\frac{1}{2}}.$$

There *are* waves with $c = 0$; this occurs (cf. (7)) for two of the four modes when $R_1^2 = R_2^2$, i.e. when $h = 0$ or $g/\alpha = 1$. This locus, however, lies entirely inside the stable region and is not adjacent to unstable waves, though it does become tangent to the stability boundary (8) as $\alpha \to 0$, reflecting the general fact that for long waves any shear layer resembles a Kelvin–Helmholtz discontinuous shear flow (cf. Drazin & Howard 1961). These stability characteristics are shown graphically in figure 1.†

In simple examples like this one it is not difficult to solve the problem completely, but with smoother (and thus perhaps physically more reasonable) profiles, such complete solutions are generally only obtainable by rather extensive numerical calculations. In such cases it is thus natural to attempt to find

† Professor J. Holmboe informed me during his recent visit to M.I.T. that he also has studied this example as well as some other similar ones, in preparation for some meteorological investigations. Another interesting example can be found in Holmboe (1962).

only the boundary of stability, and when U is odd and β even this has been done in several cases by assuming that this stability boundary is a neutral curve with $c = 0$; in fortunate cases neutral solutions with $c = 0$ can be found analytically, more or less by inspection, and in any case the numerical calculation of the locus in the Richardson-number–wave-number plane corresponding to neutral waves

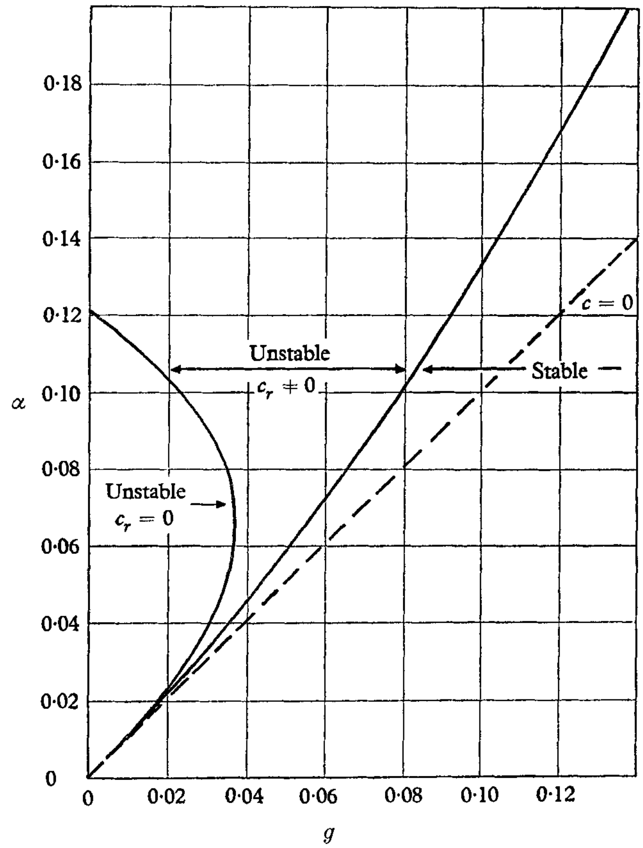

FIGURE 1. Stability characteristics of stratified shear flow.

with $c = 0$ is much easier than a complete calculation. The present example, however, serves as a warning that in some cases neutral curves found in this way may not be stability boundaries; if in the present case we had assumed the validity of the principle of exchange of stabilities and looked only for neutral waves with $c = 0$, we should have erroneously concluded that the stability boundary was $g = \alpha$. This example was in fact constructed in an attempt to understand an example with smooth profiles ($U = |y|^{\frac{1}{2}(1+a)} \operatorname{sgn} y$, $\beta = a|y|^{a-1}$ for $|y| < 1$; $U = \operatorname{sgn} y$, $\beta = 0$ for $|y| > 1$, $a > 1$) communicated to me by Miles, in which a neutral curve with $c = 0$ could be obtained. However, it lay partially inside a region known to be definitely stable from the general theorem which assures stability if $g\beta - \frac{1}{4}U'^2 > 0$ (cf. Miles 1961 and Howard 1961), and consequently could not be the stability boundary. While this example of Miles has not been analysed completely, it seems safe to conclude that its neutral curve with

$c = 0$ is analogous to $g = \alpha$ in the present example, and that the true stability boundary marks the onset of a pair of waves which are travelling, as well as growing. One should not expect a model with discontinuous profiles to be satisfactory when perturbations of *short* wavelength (compared to the natural length scale of the smooth profiles being modelled) are considered, but physical intuition as well as experience with the use of discontinuous profiles in other stability problems (cf. Drazin & Howard 1962) suggest that a reasonable resemblance with regard to stability characteristics may be anticipated for small α, at least when the effect of boundaries is negligible.

Some physical insight into the reasons for the occurrence of instability with c not pure imaginary can be obtained by examining the eigenfunction in more detail in the case of large α. Large α is a favourable case for such physical interpretation because then the disturbance is localized near the two interfaces (cf. (3)) and it becomes possible to regard the flow as two weakly coupled Kelvin–Helmholtz flows; the instability can then be regarded as arising more or less locally at each of the interfaces. An ordinary Kelvin–Helmholtz instability of wave-number α first arises (on the 'Boussinesq' model, with $\beta = \delta$) when g/α is decreased to half the square of the velocity difference across the interface, and the disturbance propagates at the average of the velocities on the two sides. Thus for large α we should expect one mode localized near the upper interface with $c = \frac{1}{2}$ on the stability boundary $g/\alpha = \frac{1}{2}$, and another localized near $y = -1$, with $c = -\frac{1}{2}$ and the same stability boundary. In fact, for large α (8) does give $g/\alpha = \frac{1}{2}$ as the stability boundary, and on it, since $a \simeq 1$, we have $c = \pm\frac{1}{2}$. With $c = +\frac{1}{2}$ and $g/\alpha = \frac{1}{2}$, the first of the equations (4) gives $A[-2]+Be^{-2\alpha}[-\frac{3}{2}] = 0$, or $A = -\frac{3}{4}Be^{-2\alpha}$. Thus $F(1) = B[1-\frac{3}{4}e^{-4\alpha}]$, $F(-1) = B\frac{1}{4}e^{-2\alpha}$, and $F(1) \gg F(-1)$. Similarly we find $F(1) \ll F(-1)$ when $c = -\frac{1}{2}$, thus verifying the interpretation of the two waves as being essentially ordinary instabilities on the separate interfaces. When α is not large, this interpretation cannot be taken so literally because the interaction of the two interfaces is more pronounced, but the same physical picture seems more or less appropriate, and one should be conscious at least of the *possibility* that $c = 0$ does not give the stability boundary whenever an antisymmetric flow is built up out of two reasonably separated shear layers.

3. Perturbation about the neutral curve

The examples of Drazin and Holmboe (cf. Miles 1961) are illustrations of cases in which it has been possible to find closed-form solutions of the stability equation by choice of special values of c (in these cases, zero). When this is possible, one obtains neutral solutions, which may be presumed to correspond to the stability boundary. The example of § 2, however, emphasizes that this is a presumption which should if possible be tested by examining the stability characteristics near the neutral curve, though it is true that neither Drazin's nor Holmboe's example seems at all likely, on the basis of the physical interpretation of the travelling disturbance of § 2, to have $c \neq 0$ on the stability boundary, and in both cases the presumed interval of instability does reduce to the known results for Richardson number $\rightarrow 0$. Miles (1963) shows how (in the analytic case) the behaviour near the neutral curve can be determined; but it seems desirable to obtain a more or less explicit *formula* analogous to the one obtained by Lin (1955) in the constant-density case which enables one to compute $\partial c/\partial \alpha^2$ at the neutral point, once the neutral solution is known. While perhaps not entirely satisfactory from the point of view of mathematical rigour, the following derivation (which is related to the method used by Lin), leads quite directly to such a formula.

In attempting to generalize Lin's result it is important to employ the proper dependent variable. This is in fact already clear in the homogeneous case where the variable used is the stream-function ϕ; this, rather than, say, $F\ (=\phi/(U-c))$, is the variable which does not become singular at the critical point as neutral stability is approached, because the singularity at $U = c$ in the coefficient $-U''/(U-c)$ occurring in the equation for ϕ is cancelled by the zero in U''. The situation is made simpler in the homogeneous case by the fact that the value of c at the neutral point is known to be $c_s = U_s \equiv U(y_s)$, where $U''(y_s) = 0$. In the stratified case, we may take $c = U_s$ at the neutral point but y_s then is not necessarily the inflexion point; it must itself be determined along with the neutral eigenfunction. That this is to be expected can be seen, for instance, from the equation for the stream-function ϕ, which (with neglect of inertial effects of stratification) is

$$\phi'' - \left[\alpha^2 + \frac{U''}{U-c} - \frac{g\beta}{(U-c)^2}\right]\phi = 0. \qquad (9)$$

If c is real, we have a singularity in $U''/(U-c)$ unless $U'' = 0$ where $U = c$, but the stratification term unavoidably introduces an even stronger singularity; in fact in the stratified case ϕ does acquire some singular behaviour as neutral stability is approached, as can be anticipated in the analytic case by noting (cf. Miles 1961, 1963) that the exponents at the singularity at $U = c$ are $\frac{1}{2} \pm (\frac{1}{4} - J_c)^{\frac{1}{2}}$, J_c being the value of the Richardson number $g\beta/U'^2$ at the point where $U = c$. In the analytic case Miles (1961) has pointed out that in the limit of neutral stability (i.e. for a 'singular neutral mode') the eigensolution becomes the solution associated with one of these exponents, or rather a certain definite branch of it. In the non-analytic case the analogue of this appears to be that the stream-function for a singular neutral mode is of the form $\phi = (U-c)^{1-n}H$ where n is a certain number between 0 and 1 (to be specified presently), the branch of $(U-c)^{1-n}$ is one which is continuous as c approaches the real axis from above, and H is *smooth* at y_c. This can be made plausible by showing (plausibly) the existence of such a smooth H, for a suitable value of n, as follows: the equation for H is (Howard 1961)

$$H'' + 2(1-n)\,U'W^{-1}H' - [\alpha^2 + nU''W^{-1} + \{n(1-n)\,U'^2 - g\beta\}\,W^{-2}]\,H = 0 \qquad (U-c \equiv W). \quad (10)$$

Suppose c is real and in the range of U. Then usually no solution of (10) will be smooth at y_c, but if n is chosen so that $n(1-n)\,U'^2 - g\beta = 0$ at y_c there will be one solution which is, and it will have $H'_c = \frac{1}{2}(1-n)^{-1}[nU''_c/U'_c - J'_c]\,H_c$, where the subscript c denotes evaluation at y_c and $J(y) = g\beta/U'^2$ is the local Richardson number. A real value of n between 0 and 1 (actually two of them) with the above property can be found if $J_c \leqslant \frac{1}{4}$, which is anyway necessary for a singular neutral mode (Miles 1961) and we may suppose c restricted so that this is the case. The smooth H found in this way will probably not satisfy the two boundary conditions, but by suitably adjusting the values of c and α they can presumably be satisfied, if a singular neutral mode exists at all.

These observations, together with the known results for the analytic case, appear to justify the hypothesis that as neutral stability is approached through positive values of $c_i = \operatorname{Im} c$ it is the variable H (with n determined as above) which approaches continuously the smooth (real) value of H ($= H_s$) corresponding to the singular neutral mode. We proceed to derive the perturbation formula on this basis, supposing that we have a neutral solution H_s with $c = c_s$, $\alpha = \alpha_s$, and g (or some over-all Richardson number) fixed; $n(1-n) = J_{c_s} \equiv J_s$. We shall think of this neutral solution as being approached by varying α, g being fixed, and $c \to c_s$ with $c_i > 0$. Now it is an immediate consequence of (9) that the functional (suppose $c_i > 0$)

$$I = I(c,\phi) = -\int_{y_1}^{y_2} \{\phi'^2 + [U''W^{-1} - g\beta W^{-2}]\phi^2\} dy \div \int_{y_1}^{y_2} \phi^2 dy \qquad (11)$$

is stationary with respect to variations of ϕ which vanish at the endpoints, provided ϕ is a solution of (9) and zero at the boundaries. (Note that in (11) we have ϕ^2, *not* $|\phi|^2$; the problem for complex c is not Hermitian, but is self-adjoint in the ordinary real sense and so can be related to a variational, though not minimal, problem.) Let $c = c(\alpha)$ be the eigenvalue and $\phi = \phi_\alpha$ the eigenfunction corresponding to a (real) wave-number α. Then the variation in $I\{c(\alpha), \phi_\alpha\}$ produced by a small variation in α is, to first order, entirely due to the consequent variation in $c(\alpha)$, and not at all to the variation of ϕ_α; since $I\{c(\alpha), \phi_\alpha\} = \alpha^2$ we have

$$2\alpha = c'(\alpha) \left[\frac{\partial I}{\partial c}\right]_{c=c(\alpha),\ \phi=\phi_\alpha.} \qquad (12)$$

Calculating $\partial I/\partial c$, the following formula for $c'(\alpha)$ results ($c = c(\alpha)$ and $\phi = \phi_\alpha$ are hereafter understood):

$$c'(\alpha) = 2\alpha \int_{y_1}^{y_2} \phi^2 dy \div \int_{y_1}^{y_2} [-U''W^{-2} + 2g\beta W^{-3}]\phi^2 dy. \qquad (13)$$

This holds for $c_i > 0$. Now let $\alpha \to \alpha_s$, $c \to c_s$; in doing so it is essential to take account not only of the singularities explicitly present in (13), but also of the singularity in ϕ. This is brought into explicit form by replacing ϕ by the variable $H = W^{n-1}\phi$ introduced above, with $n(1-n) = J_s$ understood, so that as $\alpha \to \alpha_s$, $H \to H_s$, a smooth real-valued function which is supposed to be known (note that the limit of $\phi = W^{1-n}H$ is in general neither smooth nor real, because the branch of $(U-c_s)^{1-n}$ is to be the limit for $c \to c_s$ from *above* of a branch of $(U-c)^{1-n}$ continuous on $[y_1, y_2]$ for $c_i > 0$); (13) is replaced by

$$c'(\alpha_s) = \lim 2\alpha \int_{y_1}^{y_2} W^{2(1-n)}H^2 dy \div \int_{y_1}^{y_2} [-U''W^{-2n} + 2g\beta W^{-1-2n}]H^2 dy. \quad (14)$$

For definiteness, suppose that

$$(U-c_s)^{-2n} = |U-c_s|^{-2n} \quad \text{if} \quad U > c_s$$

and

$$(U-c_s)^{-2n} = e^{+2\pi n i}|U-c_s|^{-2n} \quad \text{if} \quad U < c_s;$$

this is easily seen to be the limiting result of defining $(U-c)^{-2n}$ by taking

$$-\pi < \arg(U-c) < 0 \quad \text{for} \quad c_i > 0.$$

Since n lies between 0 and 1, the limiting value of the numerator of the right-hand side of (14) is obtained simply by replacing c by c_s and H by H_s, using the above definition of $(U-c_s)^{-2n}$; the determination of the limit of the denominator requires a little more investigation, since formal substitution of c_s for c produces a divergent integral. This minor difficulty can be overcome by integrating by parts. Evidently it is only the neighbourhood of the point (or points) at which $U = c_s$ that need special treatment; if $U'_s \neq 0$ it is convenient to use U as an independent variable on some interval $[a,b]$ about y_s short enough that $U'_s \neq 0$ on it. Evidently the limit of the integral over any segment on which $U \neq c_s$ can be obtained simply by replacing c by c_s. (If $U'_s = 0$, a different change of variable is required, but a similar procedure may be used; if there are several points at which $U = c_s$, surround each by an interval $[a,b]$ and consider them one by one.) For the interval $[a,b]$ we have:

(*a*) If $p < 0$,

$$\lim \int_a^b f(y)\, W^{-1-p} dy = \int_a^b f(y)\,(U-c_s)^{-1-p}\, dy.$$

This is obvious if $p \leqslant -1$, and easily shown also for $p < 0$, when the improper integral converges.

(*b*) If $p = 0$, we have the well-known result

$$\lim \int_a^b f(y)\, W^{-1} dy = i\pi f_s |U'_s|^{-1} + \int_a^b (U-c_s)^{-1} f(y)\, dy,$$

the last integral being the Cauchy principal value.

(*c*) If $p > 0$,

$$\int_a^b f(y)\, W^{-1-p} dy = \int_a^b f U'^{-1}\left(-\frac{1}{p}\right) dW^{-p} = -\frac{1}{p} f U'^{-1} W^{-p}\Big|_a^b + \frac{1}{p}\int_a^b \left(\frac{f}{U'}\right)' W^{-p} dy;$$

if $0 < p < 1$, the limit of the last integral can be obtained by (*a*); if $p = 1$ by (*b*); if $p > 1$ a sufficient number of additional integrations by parts reduces the problem to either case (*a*) or (*b*). By use of these results the limit of the denominator of (14), and so $c'(\alpha_s)$, can be obtained. The formula can be given a particularly neat form in the case of odd monotonic increasing U and even β, with (supposing the phenomenon illustrated by the example of § 2 does not occur) $c_s = 0$. In this case H^2 is even and $(U-c_s)^{-2n} = U^{-2n}$ for $y > 0$ and $(U-c_s)^{-2n} = e^{2\pi n i}|U|^{-2n}$ for $y < 0$. Thus

$$\lim \int_{-y_2}^{y_2} W^{2(1-n)} H^2 dy = (1+e^{2\pi n i}) \int_0^{y_2} U^{2(1-n)} H_s^2 dy \quad (\text{for } 0 < n < 1),$$

and

$$\lim \int_{-y_2}^{y_2} -U'' W^{-2n} H^2 dy = (1-e^{2\pi n i}) \int_0^{y_2} -U'' U^{-2n} H_s^2 dy$$

(this improper integral converges for $n < 1$ because $U''(0) = 0$). Finally

$$\int_{-y_2}^{y_2} W^{-1-2n}\beta H^2 dy = \int_{-y_2}^{y_2} \frac{\beta H^2}{U'}\frac{dW^{-2n}}{-2n} = \left[-\frac{\beta H^2}{2nU'}W^{-2n}\right]_{-y_2}^{y_2} + \frac{1}{2n}\int_{-y_2}^{y_2}\left(\frac{\beta H^2}{U'}\right)' W^{-2n}dy$$

$$\to \left[-\frac{\beta H^2 U^{-2n}}{2nU'}\right]_{y_2}(1-e^{2\pi n i}) + \frac{1}{2n}(1-e^{2\pi n i})\int_0^{y_2}\left(\frac{\beta H_s^2}{U'}\right)' U^{-2n}dy. \quad (15)$$

Ordinarily one would take the first term to be zero, since $H(y_2) = 0$; but in certain examples, for instance Drazin's, $U'(y_2) = 0$ also and this may produce divergence at the *upper* limit in the integral, which is compensated by the first term. This is a minor technical point caused by the fact that the variable U used implicitly in the integration by parts is inconvenient at the upper limit though appropriate near zero; the difficulty is readily overcome by taking the upper limit $< y_2$, retaining the first term above, and then evaluating the limit of the sum of both terms as the upper limit $\to y_2$. If β/U' does not become infinite at y_2 this difficulty cannot occur; we now assume this, keeping the full formula (15) available for the exceptional case. We then obtain

$$c'(\alpha_s) = 2\alpha_s i \cot(\pi n)\int_0^{y_2} U^{2(1-n)}H_s^2 dy \div \int_0^{y_2}\left\{-U''H_s^2 + \frac{1}{n}\left(\frac{g\beta H_s^2}{U'}\right)'\right\}U^{-2n}dy. \quad (16)$$

As an example we may take Holmboe's case, which Miles (1963) has studied: $U = \tanh y$, $g\beta = J\,\mathrm{sech}^2 y$, $y_2 = \infty$. One finds $\alpha_s(1-\alpha_s) = J$, $n = \alpha_s$ and $H_s = \mathrm{sech}^{\alpha_s} y$. (16) then gives, after a little elementary calculation,

$$c'(\alpha_s) = \frac{i\cos\pi\alpha_s}{\pi\alpha_s}B(\tfrac{3}{2}-\alpha_s, \alpha_s), \quad (17)$$

which can easily be shown to agree with Miles's results. It should be recalled that $c'(\alpha_s)$ is the derivative with g (or in dimensionless form with some over-all Richardson number J) held constant, and so should perhaps better be denoted by $(\partial c/\partial\alpha)_J$. This is not so useful as $(\partial c/\partial J)_\alpha$ near the point of maximum J on the neutral curve. But if the neutral curve is given by $J = J_s(\alpha)$, say, we have

$$\left(\frac{\partial c}{\partial J}\right)_\alpha = \frac{\partial(c,\alpha)}{\partial(J,\alpha)} = \frac{\partial(c,\alpha)}{\partial(c,J)}\frac{\partial(c,J)}{\partial(J,\alpha)} = -\left(\frac{\partial c}{\partial\alpha}\right)_J\left[\left(\frac{\partial J}{\partial\alpha}\right)_c\right]^{-1}.$$

Since the neutral curve is $c = 0$ (in cases like Holmboe's; otherwise we may make the same argument with c_i replacing c) $(\partial J/\partial\alpha)_c = J_s'(\alpha)$ on it, and thus we have

$$\left(\frac{\partial c}{\partial J}\right)_\alpha = -\frac{1}{J_s'}\left(\frac{\partial c}{\partial\alpha}\right)_J; \quad (18)$$

combining this with (17) one finds for Holmboe's example, after a little transformation:

$$\left(\frac{\partial c}{\partial J}\right)_\alpha = -\frac{i}{\pi\alpha_s}B(\alpha_s, \tfrac{1}{2}). \quad (19)$$

In Drazin's case, $U = \tanh y$, $g\beta = J$ (= const.), $n = \alpha^2$; applying these formulas (using the full form of (15) and letting $y_2 \to \infty$ to evaluate the denominator of (14) as mentioned above) I obtained the following results applicable on the neutral curve $J = \alpha^2(1-\alpha^2)$:

$$\left.\begin{aligned}\left(\frac{\partial c}{\partial \alpha}\right)_J &= \frac{2i\cos\pi\alpha_s^2}{\pi\alpha_s}B(\tfrac{3}{2}-\alpha_s^2,\alpha_s^2),\\ \left(\frac{\partial c}{\partial J}\right)_\alpha &= -\frac{i}{\pi\alpha_s^2}B(\alpha_s^2,\tfrac{1}{2}).\end{aligned}\right\}\tag{20}$$

It is interesting to observe that these results, like the equation for the neutral curve, are exactly the same as in Holmboe's example except that α is everywhere replaced by α^2.

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